Linear partial differential equations of high order with constant coefficients

A linear differential equation of n^{th} order with constant coefficients of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \cdots + a_n \frac{\partial^n z}{\partial y^n} +$$

$$b_0 \frac{\partial^{n-1} z}{\partial x^{n-1}} + b_1 \frac{\partial^{n-1} z}{\partial x^{n-2} \partial y} + b_2 \frac{\partial^{n-1} z}{\partial x^{n-3} \partial y^2} + \cdots + b_{n-1} \frac{\partial^{n-1} z}{\partial y^{n-1}}$$

$$+ \cdots + \ell_0 \frac{\partial^2 z}{\partial x^2} + \ell_1 \frac{\partial^2 z}{\partial x \partial y} + \ell_2 \frac{\partial^2 z}{\partial y^2} + \ell_3 \frac{\partial z}{\partial x} + \ell_4 \frac{\partial z}{\partial y} + \ell_5 z = G(x, y)$$

where $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_{n-1}, \ell_0, \ell_1, \ell_2, \ell_3, \ell_4, \ell_5$ are constants.

Homogeneous linear partial differential equations

Using the standard notation $D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}$ the above equation can be written as

$$[a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \cdots + a_n D'^n +$$
$$b_0 D^{n-1} + b_1 D^{n-2} D' + b_2 D^{n-3} D'^2 + \cdots + b_{n-1} D'^{n-1} +$$
$$+ \cdots + \ell_0 D^2 + \ell_1 DD' + \ell_2 D'^2 + \ell_3 D + \ell_4 D' + \ell_5]z = G(x, y).$$

The **homogenous equations of order** n is of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \cdots + a_n \frac{\partial^n z}{\partial y^n} + = G(x, y)$$
$$[a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 \cdots + a_n D'^n]z = G(x, y).$$

Complementary functions

To find the complementary functions for the linear homogenous partial differential equation of order n we consider

$$[a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \cdots + a_n D'^n] z = 0. \qquad (3)$$

Let us assume that

$$z = f(y + mx)$$

be a solution of the above equation. Differentiating partially with respect to x we get

$$\begin{aligned} Dz &= mf'(y + mx) \\ D^2 z &= m^2 f''(y + mx) \\ &\vdots \\ D^n z &= m^n f^{(n)}(y + mx). \end{aligned}$$

Complementary functions

Similarly differentiating partially with respect to y we get $D'^{n} z = f^{(n)}(y + mx)$. And the mixed partial derivative is given by

$$D^{n-r} D'^{r} z = m^{n-r} f^{(n)}(y + mx).$$

Substituting these values in (3) we get

$$[a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n] \;\; f^{(n)}(y + mx) = 0.$$

Since f is arbitrary $f^{(n)}(y + mx) \neq 0$. Hence

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n = 0. \tag{4}$$

This equation is known as **auxiliary equation** which is an algebraic equation of n^{th} degree in m hence by fundamental theorem of algebra it has n roots.

Complementary functions

Case (i) : If the roots are distinct (real or complex) say $m_1, m_2, \ldots, m_n$, then the complementary function is given by

$$z = f_1(y + m_1x) + f_2(y + m_2x) + \cdots + f_n(y + m_nx).$$

Case (ii) : If the r roots are equal say $m_1 = m_2 = \cdots = m_r$, then the complementary function is given by

$$\begin{aligned} z = f_1(y + m_1x) + xf_2(y + m_1x) + x^2f_3(y + m_1x) + \cdots + x^r f_r(y + m_1x) \\ + f_{r+1}(y + m_{r+1}x) + \cdots + f_n(y + m_nx). \end{aligned}$$

For $r = 2$ we have

$$z = f_1(y + m_1x) + xf_2(y + m_1x) + f_3(y + m_3x) + \cdots + f_n(y + m_nx).$$

For $r = 3$ we have

$$z = f_1(y + m_1x) + xf_2(y + m_1x) + x^2f_3(y + m_1x) + f_4(y + m_4x) + \cdots + f_n(y + m_nx).$$

Example 16.

Solve $(D^2 - 5DD' + 6D'^2)z = 0.$

Solution.

$$\begin{aligned} \textit{The auxillary equation is } m^2 - 5m + 6 &= 0 \\ (m-2)(m-3) &= 0 \\ m &= 2, 3. \\ z = f_1(y+2x) + f_2(y+3x). \end{aligned}$$

Example 17.

Solve $(D^2 - 4DD' + 4D'^2)z = 0.$

Solution.

$$\begin{aligned} \textit{The auxillary equation is } m^2 - 4m + 4 &= 0 \\ (m-z)^2 &= 0 \\ m &= 2, 2. \\ z = f_1(y+2x) + xf_2(y+2x). \end{aligned}$$

Example 18.

Solve $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0.$

Solution.

$$\text{The auxillary equation is } m^3 - 6m^2 + 11m - 6 = 0$$
$$(m-1)(m-2)(m-3) = 0$$
$$m = 1, 2, 3.$$
$$z = f_1(y+x) + f_2(y+2x) + f_2(y+2x).$$

Example 19.

Solve $(D^4 - 16D'^4)z = 0.$

Solution.

$$\text{The auxillary equation is } m^4 - 16 = 0$$
$$(m^2-4)(m^2+4) = 0$$
$$m = \pm 2, \pm 2i.$$
$$z = f_1(y+2x) + f_2(y-2x) + f_3(y+2ix) + f_4(y-2ix).$$

Example 20.

Solve $(D^4 - 2D^3D' + 2DD'^3 - D'^4)z = 0$.

Solution.

$$\begin{aligned} \textit{The auxillary equation is} \quad m^4 - 2m^3 + 2m - 1 &= 0 \\ (m^2-1)(m-1)^2 &= 0 \\ (m+1)(m-1)^3 &= 0 \\ m &= -1, 1, 1, 1. \end{aligned}$$

$$z = f_1(y-x) + f_2(y+x) + xf_3(y+x) + x^2 f_4(y+x).$$

The particular Integral

Let $F(D, D')z = G(x, y)$ be homogeneous of non-homogeneous linear partial differential equation with constant coefficients. Then the particular integral **(P.I.)** is given by

$$P.I. = \frac{1}{F(D, D')} G(x, y).$$

Case (i). If $G(x, y) = e^{ax+by}$ then the particular integral is given by

$$P.I. = \frac{1}{F(D, D')} e^{ax+by} = \frac{1}{F(a, b)} e^{ax+by}$$

provided $F(a, b) \neq 0$.

The particular Integral

If $F(a,b)=0, (D-\frac{a}{b}D')$ or its power will be a factor for $F(D,D')=0$. In this case it can be factorized and proceed as follows:

$$P.I. = \frac{1}{(D-\frac{a}{b}D')F_1(D,D')}e^{ax+by} = \frac{1}{F_1(a,b)}x\ e^{ax+by}$$

provided $F_1(a,b)\neq 0$.

$$P.I. = \frac{1}{(D-\frac{a}{b}D')^2F_2(D,D')}e^{ax+by} = \frac{1}{F_2(a,b)}\frac{x^2}{2}e^{ax+by}$$

provided $F_2(a,b)\neq 0$.

$$\vdots$$

$$P.I. = \frac{1}{(D-\frac{a}{b}D')^rF_r(D,D')}e^{ax+by} = \frac{1}{F_r(a,b)}\frac{x^r}{r!}e^{ax+by}$$

provided $F_r(a,b)\neq 0$.

Example 21.

Solve $(D^2 - 4DD' + 3D'^2)z = e^{2x+3y}$.

Solution.

$$\text{The auxillay equation is } m^2 - 4m + 3 = 0$$
$$(m-1)(m-3) = 0$$
$$m = 1, 3.$$
$$C.F = f_1(y+x) + f_2(y+3x)$$
$$P.I = \frac{1}{D^2 - 4DD' + 3D'^2} e^{2x+3y}$$
$$= \frac{1}{2^2 - 4(2)(3) + 3(3)^2)} e^{2x+3y}$$
$$= \frac{1}{4 - 24 - 27} e^{2x+3y}$$
$$= \frac{1}{7} e^{2x+3y}.$$
$$z = f_1(y+x) + f_2(y+3x) + \frac{1}{7} e^{2x+3y}.$$

Example 22.

Solve $(D^2 - D'^2)z = e^{x-y}$.

Solution.

$$\begin{aligned} \text{The auxillary equation is } m^2 - 1 &= 0 \\ (m-1)(m+1) &= 0 \\ m &= \pm 1. \end{aligned}$$

$$C.F = f_1(y+x) + f_2(y-x).$$

$$\begin{aligned} P.I. &= \frac{1}{D^2 - D'^2} e^{x-y} \\ &= \frac{1}{(D-D')(D+D')} e^{x-y} \\ &= \frac{1}{(1-(-1))(D+D')} e^{x-y} \\ &= \frac{1}{2} x \; e^{x-y}. \end{aligned}$$

$$z = f_1(y+x) + f_2(y-x) + \frac{1}{2} x \; e^{x-y}.$$

Example 23.

Solve $(D^2 - 4DD' + 4D'^2) = e^{2x+y}$.

Solution.

$$\text{The auxillary equation is } m^2 - 4m + 4 = 0$$
$$(m-2)^2 = 0$$
$$m = 2, 2.$$
$$C.F = f_1(y+2x) + xf_2(y+2x).$$

$$P.I = \frac{1}{D^2 - 4DD' + 4D'^2} e^{2x+y}$$
$$= \frac{1}{(D-2D')^2} e^{2x+y}$$
$$= \frac{x^2}{2} e^{2x+y}.$$

$$z = f_1(y+2x) + xf_2(y+2x) + \frac{x^2}{2} e^{2x+y}.$$

Example 24.

Solve $(D^3 - 5D^2D' + 8DD'^2 - 4D'^3)z = e^{2x+y}$.

Solution.

$$\text{The auxillary equation is } m^3 - 5m^2 + 8m - 4 = 0$$
$$(m-1)(m-2)(m-2) = 0$$
$$m = 1, 2, 2.$$
$$C.F = f_1(y+x) + f_2(y+2x) + xf_2(y+2x).$$
$$P.I. = \frac{1}{D^3 - 5D^2D' + 8DD'^2 - 4D'^3} e^{2x+y}$$
$$= \frac{1}{(D-D')(D-2D')^2} E^{2x+y}$$
$$= \frac{x^2}{2} e^{2x+y}.$$
$$z = f_1(y+x) + f_2(y+2x) + xf_2(y+2x) + \frac{x^2}{2} e^{2x+y}.$$

Case (ii)

If $G(x,y) = \cos(ax+by)$ or $\sin(ax+by)$ then the particular integral is given by

$$P.I. = \frac{1}{F(D,D')}\cos(ax+by)\ (OR)\ \sin(ax+by)$$
$$= R.P.\ or\ I.P. \frac{1}{F(D,D')} e^{i(ax+by)},$$

then proceed as in the **Case (i)**.

Example 25.

Solve $(D^2 - DD' - 2D'^2)z = \sin(3x + 4y)$.

Solution.

$$\begin{aligned} \textit{The auxiliary equation is } m^2 - m - 2 &= 0 \\ (m-2)(m+1) &= 0 \\ m &= 2, -1. \end{aligned}$$

$$C.F = f_1(y + 2x) + f_2(y - x).$$

$$\begin{aligned} P.I. &= \frac{1}{D^2 - DD' - 2D'^2} \sin(3x + 4y) \\ &= I.P. \frac{1}{D^2 - DD' - 2D'^2} e^{i(3x+4y)} \\ &= I.P. \frac{1}{(3i)^2 - (3i)(4i) - 2(4i)^2} e^{i(3x+4y)} \\ &= I.P. \frac{1}{-9 + 12 + 32} e^{i(3x+4y)} \\ &= I.P. \frac{1}{35} [\cos(3x + 4y) + i \ \sin(3x + 4y)] \\ &= \frac{1}{35} \quad \sin(3x + 4y). \end{aligned}$$

$$z = f_1(y + 2x) + f_2(y - x) + \frac{1}{35} \sin(3x + 4y).$$

Example 26.

Solve $(D^2 - 2DD' + D'^2)z = \cos(x - 3y)$.
Solution.

$$\begin{aligned}
\textit{The auxiliary equation is } m^2 - 2m + 1 &= 0 \\
(m-1)^2 &= 0 \\
m &= 1, 1. \\
C.F &= f_1(y+x) + xf_2(y+x).
\end{aligned}$$

$$\begin{aligned}
P.I &= \frac{1}{D^2 - 2DD' + D'^2}\cos(x-3y) \\
&= R.P. \frac{1}{D^2 - 2DD' + D'^2} e^{i(x-3y)} \\
&= R.P. \frac{1}{(i)^2 - 2(i)(-3i) + (-3i)^2} e^{i(x-3y)} \\
&= R.P. \frac{1}{-1-6-9} e^{i(x-3y)} \\
&= R.P. \frac{1}{-16}[\cos(x-3y) + i\ \sin(x-3y)] \\
&= -\frac{1}{16}\cos(x-3y).
\end{aligned}$$

$$z = f_1(y+x) + xf_2(y+x) - \frac{1}{16}\cos(x-3y).$$

Example 27.

Solve $(D^2 + 4DD' - 5D'^2)z = \sin(2x + 3y)$.
Solution.

$$
\begin{aligned}
\textit{The auxiliary equation is } m^2 + 4m - 5 &= 0\\
(m-1)(m+5) &= 0\\
m &= 1, -5.\\
C.F = f_1(y+x) + f_2(y-5x).&\\
P.I &= \frac{1}{D^2 + 4DD' - 5D'^2}\sin(2x+3y)\\
&= I.P.\frac{1}{D^2 + 4DD' - 5D'^2}e^{i(2x+3y)}\\
&= I.P.\frac{1}{(2i)^2 + 4(2i)(3i) - 5(3i)^2}e^{i(2x+3y)}\\
&= I.P.\frac{1}{-4 - 24 + 45}e^{i(2x+3y)}\\
&= I.P.\frac{1}{17}[\cos(2x+3y) + i\ \sin(2x+3y)]\\
&= \frac{1}{17}\sin(2x+3y).
\end{aligned}
$$

$$z = f_1(y+x) + f_2(y-5x) + \frac{1}{17}\sin(2x+3y).$$

Example 28.

Solve $(2D^2 - 5DD' + 2D'^2)z = 5\sin(2x+y)$.
Solution.

$$\text{The auxiliary equation is } 2m^2 - 5m + 2 = 0$$
$$(2m-1)(m-2) = 0$$
$$m = 2, \frac{1}{2}.$$
$$C.F. = f_1(y+2x) + f_2(y + \frac{1}{2}x).$$
$$P.I. = \frac{1}{2D^2 - 5DD' + 2D'^2} 5\ \sin(2x+y)$$
$$= I.P. \frac{1}{(2D-D')(D-2D')} 5e^{i(2x+y)}$$
$$= I.P. \frac{1}{(2(2i)-i)} 5x\ e^{i(2x+y)}$$
$$= I.P. \frac{-i}{3} 5x[\cos(2x+y) + i\ \sin(2x+y)]$$
$$= -\frac{5}{3}x\ \cos(2x+y).$$

$$z = f_1(y+2x) + f_2(y + \frac{1}{2}x) - \frac{5}{3}x\ \cos(2x+y).$$

Example 29.

Solve $(D^3 + D^2D' - DD'^2 - D'^3)z = e^x \cos(2y)$.

Solution.

$$\begin{aligned} \text{The auxillary equation is } m^3 + m^2 - m - 1 &= 0 \\ m^2(m+1) - (m+1) &= 0 \\ (m^2-1)(m+1) &= 0 \\ m &= 1, -1, -1. \\ C.F = f_1(y+x) + f_2(y-x) + xf_3(y-x). \end{aligned}$$

$$\begin{aligned} P.I. = \frac{1}{D^3 + D^2D' - DD'^2 - D'^3} e^x \cos(2y) &= R.P \frac{1}{D^3 + D^2D' - DD'^2 - D'^3} e^x e^{i2y} \\ = R.P \frac{1}{(1)^3 + (1)^2(2i) - (1)(2i)^2 - (2i)^3} e^{x+i2y} &= R.P. \frac{1}{1+2i+4+8i} e^{x+i2y} \\ = R.P. \frac{1}{5(1+2i)} e^{x+i2y} = R.P. \frac{1}{5(1+2i)} \frac{1-2i}{1-2i} e^{x+i2y} &= R.P. \frac{1-2i}{5(1+4)} e^x e^{i2y} \\ = R.P. \frac{1-2i}{25} e^x [\cos(2y) + i\sin(2y)] &= \frac{e^x}{25}[\cos(2y) + 2\sin(2y)]. \end{aligned}$$

$z = f_1(y+x) + f_2(y-x) + x\ f_3(y-x) + \frac{e^x}{25}(\cos 2y + 2\sin\ 2y)$.

Example 30.

Solve $(D^3 + D^2D' - DD'^2 - D'^3)z = \cos(2x + y)$.

Solution. *The complementary function is* $f_1(y - x) + x\ f_2(y - x) + f_3(y + x)$.

$$\begin{aligned}
P.I &= \frac{1}{D^3 + D^2D' - DD'^2 - D'^3} \cos(2x + y) \\
&= R.P. \frac{1}{D^3 + D^2D' - DD'^2 - D'^3} e^{i(2x+y)} \\
&= R.P. \frac{1}{(2i)^3 + (2i)^2(i) - (2i)(i)^2 - (i)^3} e^{i(2x+y)} \\
&= R.P. \frac{1}{-8i - 4i + 2i + i} e^{i(2x+y)} \\
&= R.P. \frac{1}{-9i} e^{i(2x+y)} \\
&= R.P. \frac{i}{9} [\cos(2x + 3y) + i\ \sin(2x + y)] \\
&= -\frac{1}{9} \sin(2x + y).
\end{aligned}$$

$$z = f_1(y - x) + x\ f_2(y - x) + f_3(y + x) - \frac{1}{9} \sin(2x + y).$$

Example 31.

Solve $(D^3 + D^2D' - DD'^2 - D'^3)z = \cos(x+y)$.

Solution.

$$\begin{aligned} \textit{The auxillary equation is } m^3 + m^2 - m - 1 &= 0 \\ m^2(m+1) - (m+1) &= 0 \\ (m^2-1)(m+1) &= 0 \\ (m^2-1)(m+1) &= 0 \\ m &= 1, -1, -1. \\ C.F = f_1(y+x) + f_2(y-x) + x\ f_3(y-x). \end{aligned}$$

$$P.I = \frac{1}{D^3 + D^2D' - DD'^2 - D'^3}\cos(x+y) = R.P \frac{1}{(D-D')(D^2+2DD'+D'^2)} e^{i(x+y)}$$

$$= R.P. \frac{1}{((i)^2 + 2(i)(i) + (i)^2)} x\ e^{i(x+y)} = R.P. \frac{1}{(-1-2-1)} x\ e^{i(x+y)} = R.P. \frac{1}{-4} x e^{i(x+y)}$$

$$= R.P. -\frac{1}{4} x(\cos(x+y) + i\ \sin(x+y)) = -\frac{1}{4}\ x\cos(x+y).$$

$z = f_1(y+x) + f_2(y-x) + xf_3(y-x) - \frac{1}{4}\ x\cos(x+y)$.

Case(iii).

If $G(x,y) = x^r y^s$, then the particular integral is given by

$$P.I = \frac{1}{F(D,D')} x^r y^s = [FD, D']^{-1} x^r y^s,$$

Now expand $[F(D,D')]^{-1}$ as a binomial series and operate on $x^r y^s$.

Example 32.

Solve $(D^2 - 2DD')z = x^3 y$.

Solution. *Complementary function is* $F = f_1(y) + f_2(y+2x)$.

$$P.I = \frac{1}{D^2 - 2DD'} x^3 y = \frac{1}{D^2\left[1 - \frac{2D'}{D}\right]} x^3 y = \frac{1}{D^2}\left[1 - \frac{2D'}{D}\right]^{-1} x^3 y$$

$$= \frac{1}{D^2}\left[1 - \frac{2D'}{D} + \frac{4D'^2}{D^2} + \cdots\right] x^3 y = \frac{1}{D^2}\left[1 - \frac{2D'}{D} + \frac{4D'^2}{D^2}\right]^{-1} x^3 y$$

$$= \frac{1}{D^2}\left[x^3 y + \frac{2}{D} x^3 + 0\right] = \frac{1}{D^2}\left[x^3 y + \frac{2x^4}{4} + 0\right] = \frac{x^5 y}{4 \times 5} + \frac{x^6}{2 \times 5 \times 6} = \frac{x^5 y}{20} + \frac{x^6}{60}.$$

$z = f_1(y) + f_2(y+2x) + \frac{x^5 y}{20} + \frac{x^6}{60}$.

Example 33.

Solve $(D^2 + 2DD' + D'^2)z = x^2 + xy - y^2$.
Solution. *The complementary function is* $f_1(y - x) + x\ f_2(y - x)$.

$$P.I = \frac{1}{D^2 + 2DD' + D'^2}(x^2 + xy - y^2) = \frac{1}{D^2\left[1 + \frac{2D'}{D} + \frac{D'^2}{D^2}\right]}(x^2 + xy - y^2)$$

$$= \frac{1}{D^2}\left[1 + \frac{2D'}{D} + \frac{D'^2}{D^2}\right]^{-1} x^2 + xy - y^2$$

$$= \frac{1}{D^2}\left[1 - \frac{2D'}{D} - \frac{D'^2}{D^2} + \frac{4D'^2}{D^2} + \cdots\right] x^2 + xy - y^2$$

$$= \frac{1}{D^2}\left[x^2 + xy - y^2 - \frac{2}{D}(x - 2y) + 3\frac{1}{D^2}(-2)\right]$$

$$= \frac{1}{D^2}[x^2 + xy - y^2 - x^2 + 4xy - 3x^2]$$

$$= \frac{1}{D^2}[5xy - y^2 - 3x^2]$$

$$= \left[\frac{5}{6}x^3y - \frac{1}{2}x^2y^2 - \frac{1}{4}x^4\right].$$

$$z = f_1(y - x) + x\ f_2(y - x) + \frac{5}{6}x^3y - \frac{1}{2}x^2y^2 - \frac{1}{4}x^4.$$

Case (iv)

If $G(x,y) = e^{ax+by}x^r y^s$ or $\cos\ ax + by\ x^r y^s$ or $\sin ax + by\ x^r y^s$ the particular integral is given by

$$\begin{aligned} P.I. &= \frac{1}{F(D,D')}e^{(ax+by)}x^r y^s = \frac{e^{(ax+by)}}{F(D+a,D'+b)}x^r y^s \\ &= e^{(ax+by)}[F(D+a,D'+b)]^{-1}x^r y^s. \end{aligned}$$

Expand $[F(D+a.D'+b)]^{-1}$ as a binomial series and operate on $x^r y^s$.

$$\begin{aligned} P.I. &= \frac{1}{F(D,D')}\cos^{(ax+by)}x^r y^s = R.P.\frac{1}{F(D,D')}e^{i(ax+by)}x^r y^s \\ &= R.P.\frac{e^{i(ax+by)}}{F(D+ai,D'+bi)}x^r y^s \\ &= R.P.e^{i(ax+by)}[F(D+ai,D'+bi)]^{-1}x^r y^s. \end{aligned}$$

Expand $[F(D+ai,D'+bi)]^{-1}$ as a binomial series and operate on $x^r y^s$.

Case (iv)

$$
\begin{aligned}
P.I. &= \frac{1}{F(D,D')}\sin(ax+by)x^r y^s = \\
&I.P.\frac{1}{F(D,D')}e^{i(ax+by)}x^r y^s \\
&= I.P.\frac{e^{i(ax+by)}}{F(D+ai, D'+bi)}x^r y^s \\
&= I.P.e^{i(ax+by)}[F(D+ai, D'+bi)]^{-1}x^r y^s.
\end{aligned}
$$

Expand $[F(D+ai, D'+bi)]^{-1}$ as a binomial series and operate on $x^r y^s$.

Example 34.

Solve $\frac{\partial z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial x^2} = y \ cos \ x.$

Solution. *The complementary function is* $f_1(y+2x) + f_2(y-3x)$.

$$
\begin{aligned}
P.I &= \frac{1}{D^2 + DD' - 6D'^2} y \ \cos x = R.P. \frac{e^{ix}}{D^2 + DD' - 6D'^2} y \\
&= R.P. \frac{e^{ix}}{-1 + 2iD + D^2 + iD' + DD' - 6D'^2} y \\
&= R.P. \frac{e^{ix}}{-[1 - \{iD' + 2iD + D^2 + DD' - 6D'^2\}]} y \\
&= -R.P. e^{ix}[1 - (iD' + 2iD + D^2 + DD' - 6D'^2)]^{-1} y \\
&= -R.P. e^{ix}[1 - (iD' + 2iD + D^2 + DD' - 6D'^2)] y \\
&= -R.P. e^{ix}[y + iD'(y)] = -R.P. (\cos x + i \sin x)[y + i] \\
&= -y \cos x + \sin x \\
z &= f_1(y+2x) + f_2(y-3x) - y \ \cos x + \sin \ x.
\end{aligned}
$$

Example 35.

Solve $(D^2 - DD' - 2D'^2)z = (y-1)e^x$.

Solution. *The complementary function is* $f_1(y+2x) + f_2(y-x)$.

$$\begin{aligned}
P.I &= \frac{1}{D^2 - DD' - 2D'^2}(y-1)e^x \\
&= \frac{1}{D^2 - DD' - 2r^2}(y-1)e^x \\
&= \frac{e^x}{(D+1)^2 - (D+1)(D') - 2D'^2}(y-1) \\
&= \frac{e^x}{1 + 2D + D^2 - D'D - D' - 2D'^2}(y-1) \\
&= \frac{e^x}{[1 + (2D + D^2 - D' - DD' - 5D'^2)]}(y-1) \\
&= e^x[1 + (2D + D^2 - D' - DD' - 5D'^2)]^{-1}(y-1) \\
&= e^x[1 + (2D + D^2 - D' - DD' - 5D'^2)](y-1) \\
&= e^x[(y-1) + D'(y-1)] \\
&= e^x[y - 1 + 1] \\
&= ye^x. \\
z &= f_1(y+2x) + f_2(y-x) + ye^x.
\end{aligned}$$

Example 36.

Solve $(D^2 - 5DD' + 6D'^2)z = y \sin\ x$.

Solution. *The complementary function is* $f_1(y+2x) + f_2(y+3x)$.

$$\begin{aligned}
P.I. &= \frac{1}{D^2 - 5DD' + 6D'^2} y\ \sin\ x = I.P. \frac{1}{D^2 - 5DD' + 6D'^2} e^{ix} y \\
&= I.P. \frac{e^{ix}}{(D+i)^2 - 5(D+i)(D') - 6D'^2} y \\
&= I.P. \frac{e^{ix}}{-1 + 2id + D^2 - 5iD' - 5DD' - 6D'^2} y \\
&= I.P. \frac{e^{ix}}{-[1 + (5iD' - 2iD - D^2 + 5DD' + 6D'^2)]} y \\
&= I.P. - e^{ix}[1 + (5iD' - 2iD - D^2 + 5DD' + 6D'^2)]^{-1} y \\
&= I.P. - e^{ix}[1 - (5iD' - 2iD - D^2 + 5DD' + 6D'^2)] y \\
&= I.P. - e^{ix}[y - 5iD'(y)] = I.P. - (\cos x + i \sin x)[y - 5i] \\
&= 5\cos x - y \sin x. \\
z &= f_1(y+2x) + f_2(y+3x) + 5\cos x - y\sin x.
\end{aligned}$$

Example 37.

1. *Solve* $(D^2 - DD' - 20D'^2)z = e^{5x+y} + \sin(4x - y)$
2. *Solve* $(D^2 + DD' - 6D'^2)z = x^2y + e^{3x+y}$.
3. *Solve* $(D^3 + D^2D' - DD'^2 - D'^3)z = e^{2x+y} + \cos(x + y)$.
4. *Solve* $(D^2 - 2DD')z = x^3y + e^{2x}$.
5. *Solve* $(D^3 - 7DD'^2 - 6D'^3)z = \sin(x + 2y) + e^{2x+y}$.
6. *Solve* $(D^2 + 4DD' - 5D'^2)z = \sin(x - 2y) + 3e^{2x-y}$.
7. *Solve* $(D^2 - 6DD' + 5D'^2)z = e^x \sinh\ y + xy$.

Consider the equation of the form

$$(D - mD' - a)z = 0 \qquad (1)$$

where $D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial y}$. Then (1) becomes $p - mq = az$ which is a Lagrange equation. Hence the subsidiary equation is

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{az}.$$

By taking the first two ratios, we get

$$y + mx = c_1. \qquad (2)$$

By taking the first and third ratios, we have

$$\frac{dx}{1} = \frac{dz}{az} \implies \frac{z}{e^{ax}} = c_2. \qquad (3)$$

The complete solution of equation (1) is given by

$$\frac{z}{e^a x} = f(y+mx) = e^{ax} f(y+mx).$$

Now we consider the general form of non homogeneous equation as

$$(D - m_1 D' - a_1)(D - m_2 D' - a_2) \cdots (D - m_n D' - a_n) z = 0$$

whose solution is given by

$$z = e^{a_1 x} f_1(y + m_1 x) + e^{a_2 x} f_2(y + m_2 x) + \cdots + e^{a_n x} f_n(y + m_n x).$$

In the case of repeated-factors

$$(D - mD' - a)^r z = 0.$$

The solution is given by

$$z = e^{ax} f_1(y+mx) + x\ e^{ax} f_2(y+mx) + \cdots + x^{r-1} e^{ax}.$$

Example 38.

Solve $(D - 2D' - 3)(D - 3D' - 2)z = 0$.

Solution. *The given equation is* $(D - 2D' - 3)(D - 3D' - 2)z = 0$. *By comparing this equation with* $(D - m_1 D' - a_1)(D - m_2 D' - a_2)z = 0$. *Here* $a_1 = 3, m_1 = 2$ *and* $m_2 = 3$.

$$z = e^{3x} f_1(y + 2x) + e^{2x} f_2(y + 3x).$$

Example 39.

Solve $(D^2 - DD' + D' - 1)z = 0$.

Solution. *The given equation is* $(D - D' + 1)(D - 1)z = 0$. *By comparing this equation with* $(D - m_1 D' - a_1)(D - m_2 D' - a_2)z = 0$ *Here* $a_1 = -1, a_2 = 1, m_1 = 1$ *and* $m_2 = 0$.

$$z = e^{-x} f_1(y + x) + e^{x} f_2(y).$$

Example 40.

Solve $(D^2 + 2DD' + D'^2 + 3D + 3D' + 2)z = e^{3x+5y}$.

Solution. *The given equation is* $(D + D' + 1)(D + D' + 2)z = 0$. *By comparing this equation with* $(D - m_1D' - a_1)(D - m_2D' - a_2)z = 0$. *Here* $a_1 = -1, a_2 = -2, m_1 = -1$ *and* $m_2 = -1$.

$$C.F = e^{-x} f_1(y - x) + e^{-2x} f_2(y - x).$$

$$\begin{aligned} P.I &= \frac{1}{(D + D' + 1)(D + D' + 2)} e^{3x+5y} \\ &= \frac{1}{(3 + 5 + 1)(3 + 5 + 2)} e^{3x+5y} \\ &= \frac{1}{90} e^{3x+5y}. \end{aligned}$$

$$z = e^{-x} f_1(y - x) + e^{-2x} f_2(y - x) + \frac{1}{90} e^{3x+5y}.$$

Example 41.

Solve $(D^2 - 2DD' + D'^2 - 3D + 3D' + 2)z = (e^{3x} + 2e^{-2y})^2$.

Solution. *The given equation can be written as*
$(D - D' - 1)(D - D' - 2)z = e^{6x} + 4e^{-4y} + 4e^{3x}e^{-2y}$. *To find C.F. compare this equation with* $(D - m_1 D' - a_1)(D - m_2 D' - a_2)z = 0$. *Here* $a_1 = 1, a_2 = 2, m_1 = 1$ *and* $m_2 = 1$.

$$C.F = e^x f_1(y + x) + e^{2x} f_2(y + x).$$

$$\begin{aligned}
P.I &= \frac{1}{(D - D' - 1)(D - D' - 2)} e^{6x} + 4e^{-4y} + 4e^{3x-2y} \\
&= \frac{1}{(D - D' - 1)(D - D' - 2)} e^{6x} + \frac{1}{(D - D' - 1)(D - D' - 2)} 4e^{-4y} \\
&\qquad + \frac{1}{(D - D' - 1)(D - D' - 2)} 4e^{3x-2y} \\
&= \frac{1}{(6 - 1)(6 - 2)} e^{6x} + \frac{1}{(-(-4) - 1)(-(-4) - 2)} 4e^{-4y} + \frac{1}{(4)(3 - (-2) - 2)} 4e^{3x-2y}. \\
&= \frac{e^{6x}}{20} + \frac{e^{-4y}}{3} + \frac{e^{3x-2y}}{3}.
\end{aligned}$$

$$z = e^x f_1(y + x) + e^{2x} f_2(y + x) + \frac{e^{6x}}{20} + 2\frac{e^{-4y}}{3} + \frac{e^{3x-2y}}{3}.$$

Example 42.

Solve $(D^2 + 2DD' + D'^2 - 2D - 2D')z = \sin(x + 2y)$.

Solution. *The given equation can be written as* $(D + D')(D + D' - 2)z = \sin(x + 2y)$. *To find C.F. compare this equation with* $(D - m_1 D' - a_1)(D - m_2 D' - a_2)z = 0$. *Here* $a_1 = a, a_2 = 2, m_1 = -1$, *and* $m_2 = -1$.

$C.F. = f_1(y - x) + e^{2x} f_2(y - x)$

$$\begin{aligned}
P.I &= \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} \sin(x + 2y) \\
&= I.P. \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} e^{i(x+2y)} \\
&= I.P. \frac{1}{i^2 + 2(i)(2i) + (2i)^2 - 2(i) - 2(2i)} e^{i(x+2y)} \\
&= I.P. \frac{1}{-1 - 4 - 4 - 2(i) - 2(2i)} e^{i(x+2y)} = I.P. - \frac{e^{i(x+2y)}}{3} \frac{1}{3 + 2(i)} \frac{3 - 2i}{3 - 2i} \\
&= I.P. - \frac{\cos(x + 2y) + i \sin(x + 2y)}{3} \frac{3 - 2i}{9 + 4} \\
&= \frac{1}{39}(2\cos(x + 2y) - 3\sin(x + 2y)).
\end{aligned}$$

$$z = f_1(yx) + e^{2x} f_2(y - x) + \frac{1}{39}(2\cos(x + 2y) - 3\sin(x + 2y)).$$